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BPS black hole degeneracies and minimal automorphic representations

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Abstract: We discuss the degeneracies of 4D and 5D BPS black holes in toroidal compactifications of M-theory or type-II string theory, using U-duality as a tool. We generalize the 4D/5D lift to include all charges in $\mathcal{N} = 8$ supergravity, and compute the exact indexed degeneracies of certain 4D 1/8-BPS black holes. Using the attractor formalism, we obtain the leading micro-canonical entropy for arbitrary Legendre invariant prepotentials and non-vanishing D6-brane charge. In particular, we find that the $\mathcal{N} = 8$ prepotential is given to leading order by the cubic invariant of E_6. This suggests that the minimal unitary representation of E_8, based on the same cubic prepotential, underlies the microscopic degeneracies of $\mathcal{N} = 8$ black holes. We propose that the exact degeneracies, at fixed 4-dimensional angular momentum, are given by the Wigner function of the $E_8(\mathbb{Z})$ invariant vector in this automorphic representation. A similar conjecture relates the degeneracies of $\mathcal{N} = 4$ black holes to the minimal unipotent representation of $\mathrm{SO}(8, 24, \mathbb{Z})$.

Keywords: Topological Strings, Black Holes in String Theory, String Duality.

Contents

1. Introduction

One of the distinct successes of string theory is to provide a statistical interpretation of the Bekenstein-Hawking entropy of a class of extremal or near-extremal dyonic black holes, in terms of manifestly unitary micro-states [1 − 4]. While this agreement was originally obtained in the limit of large electric and magnetic charges, corresponding to large horizon area in Planck units, subleading corrections to the entropy have received renewed attention recently [5 − 17] (see [18 − 22] for early studies). On the macroscopic side, the latter arise from higher-derivative interactions in the effective action [23, 24], while on the microscopic side, they depend on the fine details of the underlying quantum mechanics, including a choice of statistical ensemble. Based on a re-interpretation of the attractor mechanism [25 – 27], suitably generalized to include a class of 'F-type" interactions [18 – 21], these subleading corrections to the macroscopic entropy have been conjectured to reflect finite size corrections to the microscopic entropy in a specific "mixed" statistical ensemble [5]. Furthermore, it has become apparent that the Bekenstein-Hawking-Wald entropy may be protected from "non-F-type" contributions, at least of a particular class of BPS black holes [16, 17]. Independently of these developments, a precise connection between 4D black holes, 5D black holes and 5D black strings has begun to emerge [28 – 30], providing a new handle on the counting of black hole micro-states [31, 32].

It is therefore of interest to reconsider the entropy of BPS black holes in maximally supersymmetric theories, where U-duality [33] is expected to provide a powerful constraint on the higher-derivative terms in the effective action, as well as on the microscopic degeneracies (see [34] for a review of U-duality). The indexed degeneracies of 5D 1/4-BPS black holes in type-II string theory compactified on T^5 (or M-theory on T^6) were computed in [35], relying on the invariance under the U-duality group $E_6(\mathbb{Z})$. On the other hand, the leading Bekenstein-Hawking entropy of 1/8-BPS 4D black holes in type-II compactified on T^6 (or M-theory on T^7) is known to be controlled by the quartic invariant of E_7 [36, 27, 37]. The aim of this work is to determine the subleading corrections to this formula, and formulate a conjecture which relates the exact (indexed) degeneracies of 4D 1/8-BPS black holes to automorphic representations of the U-duality group.

A brief outline of this work is as follows. In section 2, we review some relevant facts about M-theory compactified on T^7 and T^6, with special emphasis on U-duality. In particular, we introduce an important relation (2.13) between the quartic invariant of E_7 and the cubic invariant of E_6, which plays a central rôle in the sequel.

In section 3, we combine the 4D/5D lift of [28] and the 5D counting of [35] to obtain the exact helicity supertrace Ω_8 of the micro-states of four-dimensional 1/8-BPS black holes. Based on the relation (2.13) (or rather its equivalent form (2.14)), we obtain in (3.11) a generalization of the 4D/5D lift to all charges in $\mathcal{N} = 8$ supergravity.

In section 4, we compute the micro-canonical degeneracies predicted by the conjecture in [5], for general Legendre invariant tree-level prepotentials $F_0 = I_3(X)/X^0$ and arbitrary electric and magnetic charges (including the D6-brane charge), in the semi-classical approximation. The assumption of Legendre invariance greatly simplifies the computation, and is in fact a property of the prepotentials describing homogeneous vector-multiplet moduli spaces [38]. In particular, we find that the E_7-invariant Bekenstein-Hawking entropy is correctly reproduced as a function of all charges, provided $I_3(X)$ is chosen to be the cubic invariant of E_6. This relies crucially on the relation (2.13), and in fact provides a derivation (or rationale) of eqs. (2.13),(2.14). We thus conclude that the topological amplitude in $\mathcal{N} = 8$ string theory is, to leading order, $\Psi = \exp(I_3(X)/X_0)$, where $I_3(X)$ is the cubic invariant of E_6.

At this point, we observe that this E_6-invariant prepotential also underlies the minimal unipotent representation of $E_8(\mathbb{R})$ constructed in [39, 40]. This is a unitary representation of E_8 acting on a Hilbert space $\mathcal{H}$ of functions of 29 variables, 28 of which can be understood as the 28 electric charges of $\mathcal{N} = 8$ supergravity. This suggests that the degeneracies of 4D 1/8-BPS black holes may have a hidden $E_8(\mathbb{Z})$ symmetry, upon including an extra quantum number. The idea that E_8 may act as a "spectrum generating" symmetry has been suggested in the past [41, 42, 40], and is quite natural given that black holes in 4 dimensions can be viewed as instantons in 3 euclidean dimensions, where the U-duality group is enlarged to $E_8(\mathbb{Z})$ (for an analogous reason, the entropy of 5D black rings exhibits a hidden E_7 symmetry [43]). The fact that certain partition functions have a higher degree of symmetry than expected is also familiar in toroidal string compactifications (where the product of the T-duality group $\mathrm{SO}(d,d,\mathbb{Z})$ and genus g modular group $\mathrm{Sp}(2g,\mathbb{Z})$ are embedded in a larger symplectic group $\mathrm{Sp}(2gd,\mathbb{Z})$, which is a symmetry of the partition function of the bosonic zero-modes [44]) and in membrane theory (where the product of the 1-loop modular group $\mathrm{Sl}(3,\mathbb{Z})$ and the U-duality group $E_d(\mathbb{Z})$ are embedded in a larger $E_{d+2}(\mathbb{Z})$, conjectured to be a symmetry of the BPS membrane partition function [45, 46]).

In section 5, we try and flesh out this idea. After a brief review of the general construction of minimal representations, we identify the 29 variables in the minimal representation of E_8 as the 28 electric charges together with the 4D angular momentum. By analogy with the metaplectic representation of Sl(2), which we recall in Subsection 5.3, we propose that the black hole degeneracies are given by the Wigner function of a $E_8(\mathbb{Z})$ invariant distribution in $\mathcal{H}$. As explained in [39, 47], this distribution is the measure for the non-gaussian theta series of E_8, and is the product over all primes p of the spherical vector of the representation over the p-adic numbers $\mathbb{Q}_p$. We sketch a similar conjecture for 1/4-BPS black holes in heterotic string compactified on T^6 (or type-II string theory compactified on $K3 \times T^2$), which we argue is related to the minimal representation of the 3D U-duality group $D_{16} = \mathrm{SO}(8,24)$. Finally, we suggest that the conformal quantum mechanics which underlies the minimal representation of E_8 [40, 48] may be the $\mathcal{N} = 8$ realization of the quantum cosmology / attractor flow scenario considered in [14, 15].

Admittedly, the conjectures in section 5 are rather speculative, and it would be very desirable to understand the relation with earlier proposals such as [31, 32] in the $\mathcal{N} = 4$ case, or [49, 50, 35] in the $\mathcal{N} = 8$ case. If correct, their generalization to $\mathcal{N} = 2$ may turn out to have very interesting mathematical consequences.

For completeness, in appendices A and B we discuss some possible applications of the minimal automorphic representations of $E_7(\mathbb{Z})$ and $E_6(\mathbb{Z})$ to 5D and 6D black holes, respectively.

2. Black hole entropy and U-duality

Let us start by recalling a few relevant facts about M-theory compactified on T^7. The massless spectrum in 4 dimensions consists of the graviton, 8 gravitini, 28 abelian gauge fields, 56 fermions and 70 scalars. The 70=28+35+7 scalars come from the reduction of the 11 dimensional metric g_{IJ}, 3-form C_{IJK} and 6-form E_{IJKLMN} (the dual of the 3-form in 11 dimensions) on T^7, respectively, and parameterize the symmetric space $E_7/\,\mathrm{SU}(8)$ [51]. The 28 gauge fields together with their magnetic duals transform into a 56 representation of E_7. They arise by reduction of the above 11-dimensional fields, together with $K_{I;JKLMNPQR}$, which represents the magnetic dual of the graviton [34]:[1]

$$7\, g_{\mu I}\ ,\ 21\, C_{\mu IJ}\ ,\ 21\, E_{\mu IJKLM}\ ,\ 7\, K_{I;\mu JKLMNPQ}\,. \tag{2.1}$$

The corresponding charges can be fit into two 8×8 antisymmetric matrices,

$$Q = \begin{pmatrix} [M2]^{IJ} & [KKM]^I \\ -[KKM]^I & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} [M5]_{IJ} & [KK]_I \\ -[KK]_I & 0 \end{pmatrix} \tag{2.2}$$

where $[KK]_I$ corresponds to a momentum excitation along the compact direction I, $[M2]^{IJ} = -[M2]^{JI}$ to a M2-brane wrapped on the directions IJ, $[M5]_{IJ} = -[M5]_{JI}$ to a M5-brane wrapped on all compact directions *but* IJ, and $[KKM]^I$ to a Kaluza-Klein monopole wrapped in all compact directions but I. This splitting into "electric" charges Q and "magnetic" charges P is not the usual "large volume" polarization, but it is the one that makes the Sl(8) subgroup of the E_7 symmetry manifest.

The Bekenstein-Hawking entropy of 1/8-BPS black holes is given by [36, 27, 37]

$$S_{BH,4D} = \pi \sqrt{I_4(P,Q)} \tag{2.3}$$

where $I_4(P,Q)$ is the singlet in the symmetric tensor product of four 56 of E_7, also known as the "diamond" invariant:

$$I_4(P,Q) = -\mathrm{Tr}(QPQP) + \frac{1}{4}\,(\mathrm{Tr}QP)^2 - 4\,[\mathrm{Pf}(P) + \mathrm{Pf}(Q)]\,. \tag{2.4}$$

(The pfaffian is, as usual, the square root of the determinant of an antisymmetric matrix; the choice of branch is purely conventional).

Upon reduction along direction 1 to type-IIA string theory compactified on T^6, the moduli space decomposes into

$$\frac{E_7}{\mathrm{SU}(8)} = \frac{\mathrm{Sl}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,6)}{\mathrm{SO}(6)\times\mathrm{SO}(6)} \ltimes \mathbb{R}^{32}\,. \tag{2.5}$$

The first and second factor describe the axio-dilaton $E_{234567} + iV_{234567}/g_s^2 l_s^6$ and the Narain moduli of T^6, respectively, and correspond to a $\mathcal{N}=4$ supersymmetric truncation of the spectrum. The third factor corresponds to the Ramond-Ramond gauge potentials on T^6, and transforms as a spinor representation of the T-duality group SO(6,6). The black hole charges decompose into $(2,12)\oplus(1,32)$ under $\mathrm{Sl}(2)\times\mathrm{SO}(6,6)$, corresponding to 6 Kaluza-Klein momenta $[kk]_i = [KK]_i$, 6 fundamental string windings $[F1]^i = [M2]^{1i}$, 32 wrapped D-branes $[D0] = [KK]_1, [D2]^{ij} = [M2]^{ij}, [D4]_{ij} = \epsilon_{ijklmn}[D4]^{klmn}/6 = [M5]_{ij}, [D6] = [KKM]^1$, 6 wrapped NS5-branes $[NS]_i = [M5]_{1i}$ and 6 wrapped KK5-monopoles $[kkm]^i = [KKM]^i$ (here $i,j = 2,\ldots,7$):

$$Q = \begin{pmatrix} D2^{ij} & [F1]^i & [kkm]^i \\ -[F1]^i & 0 & [D6] \\ -[kkm]^i & -[D6] & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} D4_{ij} & [NS]_i & [kk]_i \\ -[NS]_i & 0 & [D0] \\ -[kk]_i & -[D0] & 0 \end{pmatrix}. \tag{2.6}$$

[1] $K_{I;JKLMNPQR}$ transforms as $\Lambda^1 \otimes \Lambda^8$ under Sl(11), and is best thought of as the multiplet of Kaluza-Klein gauge fields $g_{I\mu}$ after reduction to 3 dimensions.

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The $\mathcal{N}=4$ truncation keeps the string winding, momenta, NS5-brane and KK5-monopoles, but throws away the D-branes. It is easy to check that the entropy formula (2.3) reduces to the standard $\mathcal{N}=4$ answer [52, 27],

$$S = \pi\sqrt{(\vec{q}_e)^2(\vec{q}_m)^2 - (\vec{q}_e\cdot\vec{q}_m)^2} = \pi\sqrt{q^I_\alpha\, q^J_\beta\, q^K_\gamma\, q^L_\delta\, \eta_{IK}\,\eta_{JL}\,\epsilon^{\alpha\beta}\,\epsilon^{\gamma\delta}} \tag{2.7}$$

where η_{ij} is the signature (6,6) metric and $q^I_1 = ([kk]_i, [F1]^i)$, $q^I_2 = ([NS]_i, [kkm]^i)$.

It is also of interest to discuss some decompactification limits. As the direction 1 decompactifies, or the type-IIA string coupling goes to infinity, one recovers M-theory compactified on T^6. The multiplet of 4D black hole charges decomposes into $1+27+\overline{27}+1$ charges

$$Q_0 = [KK]_1\,; \qquad Q_A = \{[M2]^{ij}\,,\quad [KK]_i\,,\quad [M5]_{i1}\} \tag{2.8a}$$
$$P^0 = [KKM]^1\,; \qquad P^A = \{[M2]^{i1}\,,\quad [KKM]^i\,,\quad [M5]_{ij}\} \tag{2.8b}$$

where the 27 charges in the first line correspond to 5D black holes, while the $\overline{27}$ charges in the second line correspond to 5D black strings (or dipole charges). The entropy of the 5D black holes is given by [26, 49]

$$S_{BH,5D} = 2\pi\sqrt{I_3(Q_A) - (J^3_L)^2} \tag{2.9}$$

where I_3 is the cubic invariant of E_6,

$$I_3(Q_A) = \mathrm{Pf}\left([M2]^{ij}\right) + \frac{1}{5!}\epsilon_{jklmnp}[KK]_i[M2]^{ij}[M5]^{klmnp} \tag{2.10}$$

and J^3_L is the angular momentum in 5 dimensions.

Due to the symmetry of the extended Dynkin diagram of E_7, there is however a different decompactification limit which also preserves a $E'_6(\mathbb{Z})$ symmetry: this is the large volume limit of type-IIA compactified on T^6. The black hole charges again split into $1+27+\overline{27}+1$ charges

$$q_0 = [D0]\,; \qquad q_A = \{[D2]^{ij}\,,\quad [kk]_i\,,\quad [NS]_i\} \tag{2.11a}$$
$$p^0 = [D6]\,; \qquad p^A = \{[D4]_{ij}\,,\quad [F1]^i\,,\quad [kkm]^i\}\,. \tag{2.11b}$$

The two lines are exchanged under T-duality along all directions in T^6. Out of the 27 charges in (2.11), it is possible to construct a cubic invariant under E'_6,

$$I_3(q_A) = \mathrm{Pf}([D2]^{ij}) + [kk]_i[D2]^{ij}[NS]_j\,. \tag{2.12}$$

A central observation for the sequel is that, in the large volume basis (2.11), the E_7 quartic invariant (2.4) can be expressed in terms of the E'_6 cubic invariant as follows:[2]

$$I_4(P,Q) = 4p^0 I_3(q_A) - 4q_0 I_3(p^A) + 4\frac{\partial I_3(q_A)}{\partial q_A}\frac{\partial I_3(p^A)}{\partial p^A} - (p^0 q_0 + p^A q_A)^2 \tag{2.13}$$

[2]This relation is in fact known to arise in Freudenthal's triple system construction of exceptional groups, see e.g. eq. (2.15) in [41] and references therein.

where q_A and p^A $(A = 1, \ldots, 27)$ are the 27 and $\overline{27}$ multiplets in (2.11a), (2.11b), and q_0, p^0 are the D0 and D6-brane charge. This equation may be easily checked by explicit computation, but, as we shall demonstrate in section 4.2, it is a general consequence of the invariance of the $\mathcal{N} = 8$ prepotential $F_0 = I_3(X)/X^0$ under Legendre transform. It is also usefully rewritten as

$$I_4(P, Q) = \frac{1}{(p^0)^2} \left[4 I_3(Q_A) - (2 I_3(p^A) + p^0 p^I q_I)^2 \right] \tag{2.14}$$

where the sum over I runs from 0 to 27, and, intentionally using the same notation as in (2.8), $Q_A = p^0 q_A + \partial_A I_3(p^A)$. As we shall see in section 3.2, this version of the identity plays an important rôle in generalizing the 4D/5D lift of [29] to all charges of $\mathcal{N} = 8$ supergravity.

Finally, let us discuss the $\mathcal{N} = 2$ truncation of this theory. It is well known that the $\mathcal{N} = 8$ gravity multiplet splits into 1 $\mathcal{N} = 2$ gravity multiplet, 6 $\mathcal{N} = 2$ gravitini multiplets, 15 vector multiplets and 10 hypermultiplets [53]. We are interested in a truncation which preserves the general structure of type-IIA compactifications on a Calabi-Yau three-fold $\mathcal{X}$, where vector multiplets arise from two-cycles in $H_2(\mathcal{X})$. Since $H_2(T^6) = \mathbb{Z}^{15}$, we are interested in a truncation which keeps only the 15 vector multiplets. The resulting vector moduli space is $SO^*(12)/S(\mathrm{U}(6) \times \mathrm{U}(1))$, with prepotential

$$F_0(X^0; X^{IJ}) = \frac{\mathrm{Pf}(X)}{X^0} \tag{2.15}$$

where X^{IJ} is a 6×6 antisymmetric matrix[3] [53]. The cubic polynomial $\mathrm{Pf}(X)$ is recognized as the cubic intersection form on T^6. The corresponding electric and magnetic charges are the 32 D-brane charges $[D0], [D2]^{ij}, [D4]_{ij}, [D6]$, transforming as a spinor of SO(6, 6). Setting $[kk] = [kkm] = [F1] = [NS] = 0$, the entropy formula (2.3) truncates to $S_{BH;4D} = \pi\sqrt{\tilde{I}_4}$ where

$$\tilde{I}_4 = 4[D6]\mathrm{Pf}([D2]) - 4[D0]\mathrm{Pf}([D4]) + 4\mathrm{Tr}([D2][D4][D2][D4]) - ([D0][D6] + [D2][D4])^2 \,, \tag{2.16}$$

which is recognized as the singlet in the symmetric tensor product of 4 spinor representations of SO(6, 6). It is worth mentioning that formula (2.13) still holds upon replacing I_4 by $\tilde{I}_4$ and I_3 by $\tilde{I}_3(q) = \mathrm{Pf}(X)$.

For completeness, note that more drastic $\mathcal{N} = 2$ truncations with 9,6 or 1 vector multiplet(s) are also possible. The corresponding manifolds are described by a prepotential $F_0 = I_3(X)/X^0$ where $I_3(X)$ is the determinant of a 3×3 matrix (corresponding to the symmetric space $\mathrm{SU}(3,3)/S(\mathrm{U}(3) \times \mathrm{U}(3))$), a 3×3 symmetric matrix (corresponding to the symmetric space $\mathrm{Sp}(6)/S(\mathrm{U}(3) \times \mathrm{U}(1))$), or simply the third power of a single variable [53].

Since the moduli space is not corrected due to $\mathcal{N} = 8$ supersymmetry, the tree-level prepotential (2.15) is in fact exact. Note also that the higher genus topological amplitudes $R^2 F^{2h-2}$ vanish. However, it is conceivable that higher-derivative $R^4 H^{4h-4}$ interactions, computed by the $\mathcal{N} = 4$ topological string [54], may contribute to the topological amplitude in the $\mathcal{N} = 8$ setting.

[3] Equivalently, the 15 complex moduli may be fit into a 3×3 hermitian matrix with quaternionic coefficients, whose determinant is equal to $\mathrm{Pf}(X)$.

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3. Exact degeneracies of 1/8-BPS states

In this section, we combine the 4D/5D lift of [29] with the degeneracies of 5D black holes computed in [35] to derive the exact (indexed) degeneracies of a class of 4D 1/8-BPS black holes.

3.1 1/8-BPS states in II/T^5

Let us start by reviewing the result of [35], who computed a particular index

$$\Omega_{5D} = \mathrm{Tr}(-1)^{2J_L^3-2J_R^3}(2J_3^R)^2 \tag{3.1}$$

in the Hilbert space of BPS black holes in type-IIB string theory compactified on $T^5 = T^4 \times S^1$, with fixed electric charges $Q_A \in 27$ and angular momentum $\ell = 2J_3^L$. By a U-duality rotation, one may choose the standard configuration of Q_1 D1-branes wrapping S^1, Q_5 branes wrapping $S^1 \times T^4$ and N units of momentum along the circle S_1. By analysing the generalized elliptic genus of $Hilb(T^4)$, the authors of [35] conjectured the relation

$$\Omega_{5D}(N, Q_1, Q_5, \ell) = \sum_{s|(NQ_1,NQ_5,Q_1Q_5,\ell);s^2|NQ_1Q_5} s\, N(s)\, \hat{c}\left(\frac{NQ_1Q_5}{s^2}, \frac{\ell}{s}\right) \tag{3.2}$$

where $N(s)$ is the number of divisors of

$$N, Q_1, Q_5, s, \frac{NQ_1}{s}, \frac{NQ_5}{s}, \frac{Q_1Q_5}{s}, \frac{NQ_1Q_5}{s^2}, \tag{3.3}$$

$\hat{c}(n, l)$ are the Fourier coefficients of the weak Jacobi form

$$-\frac{\theta_1^2(z,\tau)}{\eta^6} := \sum_{n=0}^{\infty} \sum_{l\in\mathbb{Z}} \hat{c}(n,l) q^n y^l . \tag{3.4}$$

This formula was rigorously established for N, Q_1, Q_5 coprime, and is manifestly invariant under the subgroup of $E_6(\mathbb{Z})$ which permutes N, Q_1, Q_5.

Since Z is a weak Jacobi form of weight -2 and index 1, the Fourier coefficients are function of a single variable,

$$\hat{c}(n, l) = \hat{c}(4n - l^2) \tag{3.5}$$

with $\hat{c}(-1) = 1, \hat{c}(0) = -2, \hat{c}(1) = 8, \hat{c}(4) = -12, \ldots$. In fact, the generating function of these coefficients is a simple modular form

$$\Phi(\tau) = \sum_{n=-1}^{\infty} \hat{c}(n) q^n = \frac{\theta_4(2\tau)}{\eta^6(4\tau)} = \frac{2^4}{\theta_2^4(2\tau)\theta_3(2\tau)} \tag{3.6}$$

whose Fourier coefficients can be approximated to great accuracy by the Rademacher formula [55, 13]. Restricting for simplicity to the case where N, Q_1, Q_5, ℓ are coprime, we find

$$\Omega_{5D} \sim \hat{I}_{7/2}\left(\pi\sqrt{4NQ_1Q_5 - \ell^2}\right), \tag{3.7}$$

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up to computable exponentially suppressed corrections. Using the usual asymptotic expansion of the modified Bessel function $\hat{I}_\nu(z)$ [13], we find

$$\ln \Omega_{5D} = 2\pi\sqrt{NQ_1Q_5 - J_L^2} - 4\log(NQ_1Q_5 - J_L^2) + \cdots . \tag{3.8}$$

In particular, this formula predicts an infinite number of subleading corrections to the tree-level Bekenstein-Hawking entropy (2.9). It would be interesting to relate these corrections to higher-derivative couplings in the effective action such as R^4.

3.2 From 4D to 5D black holes

We now apply the relation between 4D and 5D black holes established recently in [28]: a 4D black hole in IIA/CY with charges $[D6], [D2]_{ij}, [D0]$ and but no D4-charge is equivalent to a 5D black hole with M2-brane charge $[M2]^{ij} = [D6][D2]^{ij}$ and angular momentum $2J_L^3 = [D6]^2[D0]$, at the tip of a Taub-NUT gravitational instanton with charge $p^0 = [D6]$. Since the geometry at the tip is locally $\mathbb{R}^4/\mathbb{Z}_{p^0}$, the Bekenstein-Hawking entropy of the 4D black hole (2.3) should be given by $1/p^0$ times the Bekenstein-Hawking entropy (2.9) of the 5D black holes. Indeed, for the above choice of charges,

$$S_{BH;4D} = 2\pi\sqrt{[D6]\mathrm{Pf}([D2]) - \frac{1}{4}([D0][D6])^2} = \frac{2\pi}{p^0}\sqrt{\mathrm{Pf}([M2]) - J_L^2} = \frac{1}{p^0}S_{BH;5D} . \tag{3.9}$$

As a matter of fact, this observation can be generalized to the $\mathcal{N}=8$ setting, by using the identity (2.14) to rewrite the 4D black hole entropy as

$$S_{BH;4D} = \frac{\pi}{|p^0|}\sqrt{4I_3(Q_A) - (2I_3(p^A) + p^0p^Iq_I)^2} . \tag{3.10}$$

The Bekenstein-Hawking of the 4-dimensional black hole is thus equal to $1/p^0$ times the Bekenstein-Hawking entropy of a 5-dimensional black hole (2.9) provided the charges are identified as

$$\begin{aligned} Q_A &= p^0q_A + \partial_A I_3(p) , && (3.11\text{a}) \\ 2J_L &= (p^0)^2q_0 + p^0p^Aq_A + 2I_3(p) . && (3.11\text{b}) \end{aligned}$$

In more detail,

$$\begin{aligned} [M2]^{ij} &= [D6][D2]^{ij} + \frac{1}{8}\epsilon^{ijklmn}[D4]_{kl}[D4]_{mn} + [F1]^i[kkm]^j - [F1]^j[kkm]^i && (3.12\text{a}) \\ [KK]_i &= [D6][kk]_i + [D4]_{ij}[kkm]^j && (3.12\text{b}) \\ [M5]_i &= [D6][NS]_i + [D4]_{ij}[F1]^j && (3.12\text{c}) \\ 2J_L^3 &= [D6]([D6][D0] + \frac{1}{2}[D4]_{ij}[D2]^{ij} + [NS]_i[F1]^i + [kk]_i[kkm]^i) . && (3.12\text{d}) \end{aligned}$$

It would be interesting to support this algebraic observation by a construction of the actual supergravity solutions.

For $[D6] > 1$, the orbifold singularity at the tip of the cigar implies that the 4D black hole will have additional twisted micro-states compared to the 5D one, which will affect subleading corrections to the entropy. For $[D6] = 1$ however, one can assume that these effects are absent and directly obtain the exact degeneracies of 4D black holes from the corresponding 5D black hole [29, 32].

Following [32], consider now a 4D black hole in type-II compactified on $T^6 = T^4 \times T^2$ with q_0 D0-branes, $q_1 = [D2]^{12}$ D2-branes wrapped on T^2, $q^{ab} = -q^{ba}$ D2-branes wrapped on T^4 and one unit of D6-brane charge. This lifts to a 5D black hole in M-theory on $T^4 \times T^2$ with spin $J_L = q_0/2$ and M2 charge $[M2]^{ij} = (q_1, q_{ab})$. Identifying one of the circles on T^2 as the M-theory circle, this is equivalent to 5D black hole in IIA string theory compactified $T^4 \times S^1$ with q_1 F1-strings, q^{ab} D2-branes wrapping T^4 and the same spin $J_L = q_0/2$. By a sequence of T-dualities, this is mapped to the standard D1-D5-kk system in type-IIB/ $T^4 \times S^1$, with central charge, angular momentum and left-moving momentum along S^1

$$c = 6\mathrm{Pf}(q_{ab})\,, \qquad J_L^3 = \frac{1}{2}q_0\,, \qquad L_0 = q_1\,. \tag{3.13}$$

The five-dimensional index (3.1) is further identified to

$$\mathrm{Tr}'(-1)^{2J_3}(2J_3)^2 \tag{3.14}$$

where J_3 is the Cartan component of the 4-dimensional spin and Tr' denotes the trace with the center of mass multiplet factored out [29]. Reinstating the center of mass coordinates, we find that (3.14) computes the eighth helicity supertrace Ω_8 in 4 dimensions, which is the first non-vanishing supertrace for 1/8-BPS multiplets ([56], appendix G). According to (3.2), the exact indexed degeneracy is therefore

$$\Omega_8 = \sum_s s\, N(s)\, \hat{c}\left(\frac{\mathrm{Pf}(Q)}{s^2}, \frac{q_0}{s}\right) \tag{3.15}$$

where $\mathrm{Pf}(Q) = q_1 \mathrm{Pf}(q)$ and $\hat{c}(n)$ are the Fourier coefficients of the modular form in (3.6). Assuming that all charges are coprime, using (3.7), we find that the microscopic degeneracies grow as

$$\Omega_8 \sim \hat{I}_{7/2}\left(2\pi\sqrt{\mathrm{Pf}(Q) - \frac{1}{4}q_0^2}\right). \tag{3.16}$$

Using the generalized 4D/5D lift in (3.11), it is natural to conjecture that, more generally, the eighth-helicity supertrace should be given by

$$\Omega_8 = \sum_{s:\nabla_X F \in \mathbb{Z}} s\, N(s)\, \hat{c}\left(\frac{I_3(Q^A)}{s^2}, \frac{J_L}{s}\right) \tag{3.17}$$

where Q^A and J_L are given in (3.11), and $N(s)$ is the number of common divisors of X^I and $\nabla_I F_0$, where

$$F_0 = \frac{I_3(X^A)}{X^0}\,, \qquad X = (s; [D2]^{ij}, [NS]_i, [kk]_i)\,. \tag{3.18}$$

The sum over s should of course be restricted to values such that all X and $\nabla_X F_0$ be integers. In addition, s should also divide $J_L/2$. Thanks to (2.14), this proposal clearly reproduces the correct leading entropy. Unfortunately due to the existence of twisted sectors when $p^0 \neq 1$, it is unclear that the subleading contributions are correctly predicted. In section 5.3, we will formulate a conjecture which potentially predicts the exact degeneracies of all 1/8-BPS states.

4. Comparison to the topological string amplitude

In general, we expect that the subleading contributions in the microscopic entropy should be related to corrections to the macroscopic Bekenstein-Hawking entropy, due to higher-derivative interactions in the effective action. A immediate problem with this idea is that subleading corrections to the entropy are non-universal, and depend on a choice of statistical ensemble. In models with $\mathcal{N} = 2$ supersymmetry, it has been suggested that the appropriate ensemble to match the macroscopic answer should be a "mixed" ensemble where magnetic charges p^I are treated micro-canonically, whereas electric charges q_I are allowed to fluctuate at a fixed electric potential ϕ^I [5]:

$$Z = \sum_{q_I \in \Lambda_e} \Omega(p^I, q_I)\, e^{\pi q_I \phi^I} := e^{\mathcal{F}(p^I, \phi^I)} \tag{4.1}$$

where Λ_e is the lattice of electric charges in the large volume polarization. At leading order, using the $\mathcal{N} = 2$ attractor formalism one finds that the free energy $\mathcal{F}$ is expressed in terms of tree-level superpotential F via

$$\mathcal{F}(p^I, \phi^I) = -\pi \operatorname{Im} F_0(X^I) \tag{4.2}$$

where $X^I = p^I + i\phi^I$, so that the free-energy is identified as the modulus square of the topological wave function $\Psi = e^{i\pi F_0/2}$,

$$e^{\mathcal{F}(p^I, \phi^I)} = \sum_{k^I \in \Lambda_e^*} \Psi^*(p^I - i\phi^I - 2k^I)\Psi(p^I + i\phi^I + 2k^I)\,. \tag{4.3}$$

The summation over k^I on the right-hand side is necessary in order to maintain the periodicity in imaginary integer shifts of ϕ^I, as follows from the quantization condition over q_I in (4.1) [13, 57]. In particular, the Legendre transform of $\mathcal{F}(p^I, \phi^I)$ with respect to ϕ^I reproduces the leading Bekenstein-Hawking entropy. In general, there are higher contributions from worldsheet instantons and $R^2 F^{2h-2}$ higher-derivative interactions, but those are absent in $\mathcal{N} = 8$. There could be additional contributions e.g. due to R^4 couplings, but their precise form is not known at this stage. From the knowledge of the microscopic degeneracies $\Omega(p^I, q_I)$, one could in principle compute $\mathcal{F}(p^I, \phi^I)$ via (4.1). Conversely, from the latter one can obtain the microscopic degeneracies by Laplace transform,

$$\Omega(p^I, q_I) = \int d\phi^I\, e^{\mathcal{F}(p^I, \phi^I) - \pi q_I \phi^I}\,. \tag{4.4}$$

The integral (4.4) was evaluated in [13] for classical prepotentials

$$F_0 = \frac{I_3(X^A)}{X^0} \tag{4.5}$$

given by an arbitrary cubic polynomial $I_3(X^A)$, in the absence of D6-brane charge ($p^0 = 0$). In this section, we shall compute the integral (4.4) for arbitrary charges, but for cases where F_0 is invariant under Legendre transform in all variables X^0, X^A. Remarkably, this property holds in all cases of interest in this paper.

4.1 Legendre invariant prepotentials and cubic integrals

As shown in [38], homogeneous vector-multiplet moduli spaces are classified by Jordan algebras J of degree 3. In particular, their prepotential is of the form (4.5), where the homogeneous cubic polynomial $I_3(X^A)$ is the norm of J. As a consequence, F_0 is invariant under a Legendre transform with respect to all variables at once.[4] Independently, Legendre-invariant homogeneous cubic polynomials in a finite number of variables have been classified in [58] (see also [59]):

(i) $G = D_{n\geq 4} : I_3 = X^1(X^2X^3 + X^4X^5 + \cdots + X^{2n-6}X^{2n-5})$;

(ii) $G = E_6 : I_3 = \det(X)$, with X a 3×3 matrix;

(iii) $G = E_7 : I_3 = \mathrm{Pf}(X)$, with X an antisymmetric 6×6 matrix;

(iv) $G = E_8 : I_3 = X^3|_{\mathbf{1}}$, with X a $\mathbf{27}$ representation of E_6 and I_3 the singlet in the cubic power of $\mathbf{27}$;

(v) $G = B_{n\geq 3} : I_3 = X^1[(X^2)^2 + X^3X^4 + \cdots + X^{2n-5}X^{2n-4}]$;

(vi) $G = F_4 : I_3 = \det(X)$, with X a symmetric 3×3 matrix;

(vii) $G = G_2 : I_3 = X^3$, with X a single variable.

We have labeled each case by a group G, since the corresponding cubic polynomial plays a crucial rôle in the minimal unitary representation of G, as we shall review in section 5.1. Case (i) corresponds to the tree-level prepotential in $\mathcal{N} = 2$ heterotic compactifications (the $n = 4$ case corresponds to the STU model), while we already encountered cases (ii-iv) in section 1 of this paper.

The assumption of invariance under Legendre transform in all variables means that the solution to the equation $\nabla_X F_0(X) = Y$ is given by $X = \nabla_Y F_0(Y)$, i.e.

$$\begin{cases} Y^A &= -\dfrac{\partial_A I_3(X)}{X^0} \\ Y^0 &= \dfrac{I_3(X)}{(X^0)^2} \end{cases} \quad \Leftrightarrow \quad \begin{cases} X^A &= -\dfrac{\partial_A I_3(Y)}{Y^0} \\ X^0 &= \dfrac{I_3(Y)}{(Y^0)^2} \,. \end{cases} \tag{4.6}$$

[4]This was not stated in this way in [38], but follows from the axioms (M1-M5) in this paper.

G	n_v	I_3	H	H_0	κ	μ	ν
$D_{n\geq 4}$	$2n-4$	$X^1 C_{ab} X^a X^b$	$A_1 \times D_{n-2}$	D_{n-4}	$(-1)^n$	$n-1$	$(n-4)/2$
E_6	10	$\det(X)$	A_5	$A_2 \times A_2$	-1	6	$1/2$
E_7	16	$\mathrm{Pf}(X)$	D_6	A_5	1	9	1
E_8	28	$27^3\vert_1$	E_7	E_6	1	15	2
$B_{n\geq 3}$	$2n-3$	$X^1 C_{ab} X^a X^b$	$A_1 \times B_{n-2}$	B_{n-4}	$(-1)^n$	$n-1$	$(n-4)/2$
F_4	7	$\det(X)$	C_3	A_2	-8	$3/2$	$n.a.$
G_2	2	x^3	A_1	1	3	2	$n.a.$

Table 1: Data entering the construction of the minimal unipotent representation of G, of functional dimension $n_v + 1$. I_3 is a homogeneous polynomial of degree 3 in $n_v - 1$ variables, such that $F = I_3(X)/X^0$ is invariant under Legendre transformation in all n_v variables. The subgroup of $H \subset G$ acts by canonical transformations and $H_0 \subset H$ by linear transformations of the variables. κ is the numerical factor entering in the Hessian (4.9), and ν is the index of the Bessel function entering in the spherical vector f_K in (5.10). For $G = F_4, G_2$, the spherical vector does not exist, as there is no K-singlet in the minimal representation.

For X and Y related as in (4.6), we have

$$I_3(X) = \frac{[I_3(Y)]^2}{(Y^0)^3}\,, \qquad \frac{I_3(X)}{(X^0)^3} = \frac{(Y^0)^3}{I_3(Y)} \tag{4.7}$$

hence

$$F_0(X) + X^0 Y^0 + X^A Y^A = -F_0(Y)\,. \tag{4.8}$$

This implies that the classical approximation to the Fourier transform of $\exp[iF_0(X)]$ is equal to $\exp[-iF_0(Y)]$.

Let us now determine the 1-loop determinant. By explicit computation, we find that the determinant of the Hessian of the map $X \to \nabla_X F$ in (4.6) is equal to

$$\det[\nabla_{X^I}\nabla_{X^J} F_0(X)] = \begin{cases} \kappa \left(\dfrac{I_3(X)}{(X^0)^3}\right)^{(n_v+2)/3}, & G \neq D_n \\ \kappa \left(\dfrac{X^1}{X^0}\right)^{n_v-4} \left(\dfrac{I_3(X)}{(X^0)^3}\right)^2, & G = D_n\,. \end{cases} \tag{4.9}$$

where the number of complex variables n_v and the numerical factor κ can be read off in table 1. From general properties of the Legendre transform: the Hessian at $Y = \nabla_X F$ is the inverse of the Hessian at X: suppressing indices for simplicity,

$$f(x) - xf'(x) = g(f'(x)) \Rightarrow -xf''(x) = f''(x)g'(f'(x)) \Rightarrow g'(f'(x)) = x \tag{4.10}$$

Differentiating once more with respect to x indeed gives

$$g''(f'(x))f''(x) = -1 \Rightarrow g''(y) = -\frac{1}{f''(x)} \tag{4.11}$$

Thus, the determinant of the Hessian at the saddle point is

$$\det[\nabla_{X^I}\nabla_{X^J} F_0(X)]|_Y = \begin{cases} \kappa \left(\frac{I_3(Y)}{(Y^0)^3}\right)^{-(n_v+2)/3}, & G \neq D_n \\ \kappa \left(\frac{Y^1}{Y^0}\right)^{-(n_v-4)} \left(\frac{I_3(Y)}{(Y^0)^3}\right)^{-2}, & G = D_n\,. \end{cases} \tag{4.12}$$

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This may also be obtained from (4.9) using the identities (4.7), and, in the D_n case, $X^0/X^1 = -Y^1/Y^0$. This implies in the semi-classical (one-loop) approximation, for $G \neq D_n$,

$$\int dX^0 dX^A (X^0)^\alpha [I_3(X)]^\beta \exp\left[iF_0(X) + i(X^0Y^0 + X^AY^A)\right] \sim \\ \sim \kappa^{-1/2}(Y^0)^{\alpha'}[I_3(Y)]^{\beta'} \exp\left[-iF_0(Y)\right] \tag{4.13}$$

where

$$\alpha' = -2\alpha - 3\beta - \frac{(n_v+2)}{2} \tag{4.14a}$$

$$\beta' = \alpha + 2\beta + \frac{(n_v+2)}{6} \tag{4.14b}$$

In the D_n case,

$$\int dX^0 dX^A (X^0)^\alpha [I_3(X)]^\beta (X^1)^\gamma \exp\left[iF_0(X) + i(X^0Y^0 + X^AY^A)\right] \sim \\ \sim \kappa^{-1/2}(Y^0)^{\alpha'}[I_3(Y)]^{\beta'}(Y^1)^{\gamma'} \exp\left[-iF_0(Y)\right] \tag{4.15}$$

where

$$\alpha' = -2\alpha - 3\beta - \gamma - \frac{(n_v+2)}{2} \tag{4.16a}$$

$$\beta' = \alpha + 2\beta + \gamma + 1 \tag{4.16b}$$

$$\gamma' = -\gamma + \frac{(n_v-4)}{2}\,. \tag{4.16c}$$

It is easy to check that the linear transformations (4.14) and (4.16) are involutions, as it is necessary if the Fourier transform is to square to one. In [58], it was shown that, for special choices of (α, β, γ), the classical approximation is in fact exact:[5] in particular, for $\beta = 0$ and $G \neq D_n$,

$$\int dX^0 dX^A\, (X^0)^{-(n_v+2)/6} \exp\left[iF_0(X) + i(X^0Y^0 + X^AY^A)\right] \\ \sim \kappa^{-1/2}(Y^0)^{-(n_v+2)/6}\, \left(\frac{I_3(Y)}{(Y^0)^3}\right) \exp\left[-iF_0(Y)\right] \tag{4.17}$$

or, in the $G = D_n$ case,

$$\int dX^0 dX^A\, (X^0)^{(2-n_v)/2}(X^1)^{(n_v-4)/2} \exp\left[iF_0(X) + i(X^0Y^0 + X^AY^A)\right] = \\ = \kappa^{-1/2}\,(Y^0)^{-1} \exp\left[-iF_0(Y)\right]. \tag{4.18}$$

(This last identity can be checked by first doing the gaussian integral over X^a, then the integral over X^0 which yields a Dirac distribution for the remaining X^1 integral). These identities will prove very useful in evaluating the integral (4.4).

[5]The idea of the proof is to use the Mellin representation $e^{I_3/x^0} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz(-I_3/x^0)^{-z}\Gamma(z)dz$ and compute the integral over x^I in terms of generalized Gamma functions.

4.2 Classical evaluation

In this section, we evaluate the classical limit of the integral (4.4), i.e. the Legendre transform of the free energy (4.2) with respect to all electric potentials $\phi^I, I = 0, \ldots, n_v - 1$. For a prepotential F_0 given by (4.5), independently of the assumption of Legendre invariance, the free energy reads

$$\mathcal{F} = \frac{\pi}{(p^0)^2 + (\phi^0)^2} \left\{ p^0 \left[\phi^A \partial_A I_3(p) - I_3(\phi) \right] + \phi^0 \left[p^A \partial_A I_3(\phi) - I_3(p) \right] \right\} . \tag{4.19}$$

In order to eliminate the quadratic term in ϕ^A and reach a form closer to (4.17), it is convenient to change variables

$$x^A = \phi^A - \frac{\phi^0}{p^0} p^A , \qquad x^0 = \frac{[(p^0)^2 + (\phi^0)^2]}{p^0} . \tag{4.20}$$

The entropy in the mixed ensemble becomes

$$\mathcal{S} = \langle \mathcal{F}(p^I, \phi^I) - \pi q_I \phi^I \rangle_{\{\phi^I\}} \tag{4.21}$$

$$= \pi \left\langle -\frac{I_3(x)}{x^0} + \frac{\partial_A I_3(p) + p^0 q^A}{p^0} x^A + \frac{2 I_3(p) + p^0 p^I q_I}{p^0} \sqrt{\frac{x^0}{p^0} - 1} \right\rangle_{\{x^I\}} \tag{4.22}$$

where the right-hand side should be extremized with respect to all ϕ^I (recall that I runs from 0 to $n_v - 1$). In order to get rid of the square root, it is convenient to introduce an auxiliary variable t, and write

$$\mathcal{S} = \pi \left\langle -\frac{I_3(x)}{x^0} + \frac{\partial_A I_3(p) + p^0 q^A}{p^0} x^A - \frac{t}{4} \left(\frac{x^0}{p^0} - 1 \right) - \frac{(2 I_3(p) + p^0 p^I q_I)^2}{t \ (p^0)^2} \right\rangle_{\{x^I, t\}} \tag{4.23}$$

At fixed t, we recognize the Legendre transform of $F_0(x) = I_3(x)/x^0$ with respect to all variables x^I, at conjugate potentials

$$y_A = \frac{\partial_A I_3(p) + p^0 q_A}{p^0} , \qquad y_0 = -\frac{t}{4 p^0} . \tag{4.24}$$

Using the Legendre invariance of $F_0(x)$, we conclude that the result of the extremization over x^I is

$$\mathcal{S} = \pi \left\langle 4 \frac{I_3(p^0 y)}{(p^0)^2 t} - \frac{[2 I_3(p) + p^0 p^I q_I]^2}{t \ (p^0)^2} - \frac{t}{4} \right\rangle_t . \tag{4.25}$$

The extremization with respect to t leads to $t = S_0/\pi$, where

$$S_0 = \frac{\pi}{p^0} \sqrt{4 I_3(p^0 y) - [2 I_3(p) + p^0 p^I q_I]^2} \tag{4.26}$$

finally leading to the classical entropy,

$$\mathcal{S}(p^I, q_I) = S_0 . \tag{4.27}$$

It is easy to check that (4.26) is consistent with the general result in [60] — in fact, Legendre invariance is what allows to solve eq. (14) in [60] in closed form. Setting $F = I_3(X)/X^0$

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as appropriate for the $G = E_8$ case, and making use of (2.14), we find that the classical entropy S_0 in (4.26) is in fact the square root of the quartic invariant of E_7. Conversely, we find that the relation (2.14) between the quartic invariant I_4 and the cubic polynomial I_3 is a general consequence of the attractor formalism, and that the entropy of $\mathcal{N} = 8$ black holes (2.3) is controlled to leading order by the prepotential $F = I_3(X)/X^0$.

The classical entropy (4.26) can be further simplified by making use once again of the Legendre invariance of F_0. Applying (4.7) to the case $X^A = \partial_A I_3(p), X^0 = 1$, we may expand

$$I_3(\partial_A I_3(p) + p^0 q_A) = [I_3(p)]^2 + p^0 I_3(p) p^A q_A + (p^0)^2 \partial^A I_3(q) \partial_A I_3(p) + (p^0)^3 I_3(q) \quad (4.28)$$

which allows us to rewrite

$$S_0 = \pi \sqrt{4p^0 I_3(q) - 4q_0 I_3(p) + 4\partial^A I_3(q) \partial_A I_3(p) - (p^0 q_0 + p^A q_A)^2} \quad (4.29)$$

reproducing (2.13).

We conclude that, to leading order, the topological amplitude controlling the entropy of 1/8-BPS black holes is

$$\Psi(X) = \exp\left(\frac{I_3(X^A)}{X^0}\right) \quad (4.30)$$

where I_3 is the cubic invariant of E_6. 15 of the complex variables X^A/X^0 may be viewed as the Kähler classes of $H_2(\mathcal{X})$, while the remaining 12 are a subset of the Narain moduli in (2.5).

4.3 Beyond the classical limit

The leading entropy (4.26) above was the result of a tree-level saddle point approximation to the integral (4.4). It however receives quantum corrections from fluctuations around the saddle point. In addition, there may be corrections to the prepotential itself, although we have little control on them. In this section, we shall assume that F_0 is uncorrected, and compute the micro-canonical degeneracies $\Omega^{(0)}(p, q)$ which result from (4.4) under this assumption.

Performing the same change of variables as in (4.20), and introducing the auxiliary variable t by the usual Schwinger representation, the OSV integral becomes

$$\begin{aligned}\Omega^{(0)} = \int \frac{1}{4\sqrt{\pi t}} dx^0 dx^A dt \exp\Bigg[\pi\Bigg(&- \frac{I_3(x)}{x^0} + \frac{\partial_A I_3(p) + p^0 q^A}{p^0} x^A - \\ &- \frac{t}{4}\left(\frac{x^0}{p^0} - 1\right) - \frac{(2I_3(p) + p^0 p^I q_I)^2}{t\,(p^0)^2}\Bigg)\Bigg]. \end{aligned} \quad (4.31)$$

The integral over x^0, x^A is now a Fourier transform of the type (4.13), with conjugate momenta y_0, y_A given in (4.24). In the saddle point approximation, we thus get, for $G \neq D_n$,

$$\int \frac{\kappa^{-1/2}}{4\sqrt{\pi t}} dt \left(\frac{2^6 I_3(p^0 y)}{t^3}\right)^{\frac{n_v+2}{6}} \exp\left[4\pi \frac{I_3(p^0 y)}{(p^0)^2 t} - \pi \frac{[2I_3(p) + p^0 p^I q_I]^2}{t\,(p^0)^2} - \frac{\pi t}{4}\right] \quad (4.32)$$

while, for $G = D_n$,

$$\int \frac{\kappa^{-1/2}}{4\sqrt{\pi t}} dt \left(\frac{2^6 I_3(p^0 y)}{t^3}\right)\left(\frac{4(\vec{p}^2+p^0 q_1)}{t}\right)^{\frac{n_v-4}{2}} \exp\left[4\pi\frac{I_3(p^0y)}{(p^0)^2 t} - \pi\frac{[2I_3(p)+p^0p^Iq_I]^2}{t\ (p^0)^2} - \frac{\pi t}{4}\right] \tag{4.33}$$

The remaining integral over t is of Bessel type, with a saddle point at $t = S_0/\pi$. The one-loop determinant $1/\sqrt{\mathcal{S}''(t^*)} = t^{1/2}$ cancels the factor of $1/\sqrt{t}$ in front, leading to the result, for $G \neq D_n$,

$$\Omega^{(0)}(p^I, q_I) \sim [I_3(p^0y)]^{(n_v+2)/6} S_0^{-(n_v+2)/2} e^{S_0} \tag{4.34}$$

or, in the D_n case,

$$\Omega_0(p^I, q_I) \sim I_3(p^0y)(\vec{p}^2 + 2p^0q_1)^{(n_v-4)/2} S_0^{-(n_v+2)/2} e^{S_0}\,. \tag{4.35}$$

It is important to note that the pre-factors appearing in these expressions are inconsistent with U-duality,[6] indicating that the naive flat integration measure in (4.4) is inconsistent with U-duality. In order to remedy this, we may use the fact that, at the saddle point, the prefactors in (4.34),(4.35) can be expressed in terms of the magnetic charges and electric potentials,

$$I_3(p^0y) = -\frac{1}{4}|X^0|^6 \left(C_{ABC}\,\mathrm{Im}\,t^A\,\mathrm{Im}\,t^B\,\mathrm{Im}\,t^C\right)^2 \tag{4.36a}$$

$$\vec{p}^2 + 2p^0q_1 = |X^0|^2\,\mathrm{Im}\,t^a C_{ab}\,\mathrm{Im}\,t^b \tag{4.36b}$$

where

$$X^I = p^I + i\phi^I\,, \qquad t^A = \frac{X^A}{X^0} \tag{4.37}$$

This should be compared to the standard expression for the Kähler potential (see e.g. [61], eq. 9.6)

$$e^{-K} = -\frac{4}{3}|X^0|^2 C_{ABC}\,\mathrm{Im}\,t^A\,\mathrm{Im}\,t^B\,\mathrm{Im}\,t^C\,. \tag{4.38}$$

In order to cure the non-U-duality invariance of (4.34), a possible option is thus to multiply the flat integration measure in (4.4) by $e^{(n_v+3)K}/|X^0|^3$; this will remove the first factor in (4.34) while leaving the power of S_0 untouched (a similar option holds for $G = D_n$). However, there is no guarantee that higher-loop corrections would be U-duality invariant under this prescription.

A more attractive option is to use the additional relation, valid at the saddle point,

$$x^0 = \frac{I_3(p^0y)}{(p^0t^2)}\,. \tag{4.39}$$

The first factor in (4.34) can therefore be removed by multiplying the integration measure by $(p^0x^0)^{(n_v+2)/6}$. Denoting by $\Omega^{(1)}$ the result of this procedure, we have, in terms of the original variables,

$$\Omega^{(1)} \sim \int d\phi^0 d\phi^A \left[(p^0)^2 + (\phi^0)^2\right]^{-(n_v+6)/2} e^{\mathcal{F}+\pi\phi^I q_I}\,. \tag{4.40}$$

[6] For $p^0 \neq 0$ they mix the electric and magnetic charges, and the option of considering ratios at fixed electric charge, as advocated in [13], is no longer available.

According to (4.17), this has the great advantage of rendering the 1-loop approximation to the integral over x^0, x^I exact. The remaining t integral becomes

$$\Omega^{(1)} = \int \frac{\kappa^{-1/2}}{4\sqrt{\pi t}} dt\, t^{-\frac{n_v+2}{6}} \exp\left[4\pi \frac{I_3(p^0 y)}{(p^0)^2 t} - \pi \frac{[2I_3(p) + p^0 p^I q_I]^2}{t\,(p^0)^2} - \frac{\pi t}{4}\right] \tag{4.41}$$

leading to the manifestly U-duality invariant result, in the $G \neq D_n$ case,

$$\Omega^{(1)} = \hat{I}_{(n_v-1)/6}(S_0) \sim S_0^{(n_v+2)/6} e^{S_0}\,. \tag{4.42}$$

Similarly, in the D_n case, using the measure in (4.18), we get a universal result

$$\Omega^{(1)} = \hat{I}_{1/2}(S_0) \sim S_0^{-1}\, e^{S_0} \tag{4.43}$$

which agrees with the previous case for $G = D_4$. We should stress that (4.40) is only an educated guess; it however meshes well with the conjecture in the next section.

We may now compare this macroscopic prediction with the microscopic counting: setting $G = E_8, n_v = 28$ as appropriate for case (iv), we find

$$\Omega^{(1)} \sim \hat{I}_{9/2}(S_0)\,. \tag{4.44}$$

On the other hand, if we believe that the attractor formalism should only describe the 16 vector multiplets described by the prepotential (2.15), the $G = E_7, n_v = 16$ case (iii) applies, leading to

$$\Omega^{(1)} \sim \hat{I}_{5/2}(S_0) \tag{4.45}$$

where S_0 is now proportional to the square root of quartic invariant (2.16) of SO(6, 6). In either case, the index of the Bessel function differs from the microscopic counting in (3.16). This suggests that there may be logarithmic corrections to the topological amplitude, or that the appropriate generating function should have modular weight 7/2 (for $G = E_8$) or 3/2 (for $G = E_7$). We leave this discrepancy as an open problem for further investigation.

5. Black hole partition functions and theta series

In the previous section, we have demonstrated that the E_7-invariant entropy formula (2.3), including all 56 electric and magnetic charges, follows from the attractor formalism based on the prepotential (4.5) where $I_3(X)$ is the cubic invariant of E_6. On the other hand, we have mentioned that this prepotential lies at the heart of the construction of the minimal representation of E_8 [39, 40]. This suggests that $E_8(\mathbb{Z})$ may be a hidden symmetry of the partition function of 1/8-BPS black holes in 4 dimensions. In this section, we try and flesh out this conjecture.

5.1 Review of the minimal unipotent representation

Let us start by a brief review of the construction of the minimal "unipotent" representation of a simple Lie groups G in the split (i.e. maximally non compact) real form (see [39] for more details, as well as [42, 40, 62 – 64] for an equivalent approach using the formalism of Jordan algebras).

The minimal representation of a non-compact group G is the unitary representation of smallest functional dimension [65] It can be obtained by quantizing the co-adjoint orbit of smallest dimension in G, i.e. the orbit of any root in the Lie algebra of G. Without loss of generality, we consider the orbit of the lowest root $E_{-\omega}$. Under the Cartan generator $H_\omega = [E_\omega, E_{-\omega}]$, the Lie algebra of G decomposes into a graded sum of 5 subspaces,

$$G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2} \tag{5.1}$$

where $G_{\pm 2}$ are one-dimensional and correspond to the highest/lowest root. G_0 further decomposes into a commuting sum $\mathbb{R} \oplus H$, where the first summand corresponds to the Cartan generator H_ω. Since G_{-2}, G_{-1} and H commute with $E_{-\omega}$, the co-adjoint orbit of $E_{-\omega}$ is generated by the action of the $H_\omega \oplus G_{+1} \oplus E_\omega$. As any co-adjoint orbit, it carries a G-invariant Kirillov-Konstant symplectic form, which decomposes into a symplectic form on G_{+1} and a symplectic form on $H_\omega \oplus E_\omega$. This endows $G_{+1} \oplus E_\omega$ with the structure of a Heisenberg algebra, whose central element is E_ω. Furthermore, G_0 acts linearly on G_{+1}. Quantization proceeds by choosing a lagrangian submanifold C in G_{+1}, and representing the generators of G as differential operators acting on the space of functions on $\mathbb{R} \times C$, where the first factor denotes the central element of the Heisenberg algebra E_ω. A standard choice of lagrangian C is to take the orbit under G_1 of E_{β_0}, where β_0 is the root attached to the affine root on the Dynkin diagram of G [39]. Let H_0 be the subgroup of H which commutes with E_{β_0}. Parameterizing G_{+1} by coordinates $x_0, x_1, \dots x_{n_v-1}$ and momenta $p^0, p^1, \dots, p^{n_v-1}$ (where x_0 is the coordinate along E_{β_0}), one can show that this lagrangian manifold is defined by the set of equations

$$C = \left\{ (x_I, p^I) \in G_{+1} \,|\, p^I = \frac{\partial F_0}{\partial x_I} \right\}, \qquad F_0 = \frac{I_3(x_A)}{x_0} \tag{5.2}$$

where $I_3(x_A)$ is the H_0-invariant cubic polynomial built out of the x_A's (in mathematical terms, it is the relative invariant of the regular prehomogeneous vector space associated to H_0, or the norm of the Jordan algebra with reduced structure group H_0). In principle, the relation (5.2) may be rewritten a set of H-covariant homogeneous constraints on the vector (x_I, p^I). The invariance of I_3 under Legendre transform implies that C is invariant under the exchange of all x_I with p^I at once: this is precisely the action of a particular element S in the Weyl group of G (the longest element in the Weyl group of H_0). Another Weyl element A (the Weyl reflection with respect to the root β_0) acts as a $\pi/2$ rotation in the (y, x_0) plane.

The result of this procedure is a unitary representation of G in the Hilbert space $\mathcal{H}$ of functions of $n_v + 1$ variables (y, x_I). Infinitesimal generators are represented by differential operators, of which we display a subset only:

$$E_\omega = y\,, \tag{5.3}$$

$$E_{\beta_I} = y\partial_I\,, \qquad E_{\gamma_I} = i x_I \tag{5.4}$$

$$H_{\beta_0} = -y\partial_y + x_0\partial_0\,, \qquad H_\omega = -\mu - 2y\partial_y - x^I\partial_I \tag{5.5}$$

$$E_{-\beta_0} = -x_0\partial_y + \frac{i}{y^2} I_3(x^A) \tag{5.6}$$

$$E_{-\omega} = yp^2 + p(x_I p^I) + x_0 I_3(p) - \frac{p^0}{y^2} I_3(x) + \frac{1}{y}\frac{\partial I_3(x)}{\partial x_A}\frac{\partial I_3(p)}{\partial p^A} \tag{5.7}$$

where $p = i\partial/\partial y, p^A = i\partial/\partial x^A$ and μ is a numerical constant displayed in table 1. In the last equation, we dropped the ordering terms for simplicity. The Weyl reflections S and A are represented as

$$(S \cdot f)(y, x_I) = \int dy_0 dy_A \; e^{i(x_I y^I)/y} f(y, y^I) \tag{5.8}$$

$$(A \cdot f)(y, x_0, x_A) = e^{-\frac{I_3(x_A)}{x_0 y}} f(-x_0, y, x_A)\,. \tag{5.9}$$

A vector of particular interest in the Hilbert space $\mathcal{H}$ is the spherical vector f_K, i.e. a function invariant under the maximal compact subgroup K of G. The spherical vector has been computed for all simply laced groups G in the split form in [39], and reads (for $G \neq D_n$)

$$f_K(\tilde{X}) = \frac{1}{|z|^{2\nu+1}} \hat{K}_\nu(S_1) e^{-iS_2} \tag{5.10}$$

where $z = y + ix_0$, $\hat{K}_\nu(x)$ is related to the modified Bessel function by $\hat{K}_\nu(x) = x^{-\nu} K_\nu(x)$, ν can be read off in table 1, and

$$S_1 = \sqrt{\sum_{\alpha=0}^{n_v} \left[\tilde{X}_\alpha^2 + (\nabla_\alpha \tilde{F})^2\right]}\,, \qquad S_2 = \frac{x_0\; I_3(x)}{y(y^2 + x_0^2)} \tag{5.11}$$

and $\tilde{X} = (x_0, x_A, y)$ and

$$\tilde{F}_0(\tilde{X}) = \frac{I_3(x_A)}{\sqrt{y^2 + x_0^2}}\,. \tag{5.12}$$

For $G = D_n$, the spherical vector is instead

$$f_K(\tilde{X}) = |z|^{-1} \left(1 + \frac{x_1^2}{|z|^2}\right)^{(n-4)/2} \hat{K}_{(n-4)/2}(S_1) e^{-iS_2}\,. \tag{5.13}$$

The term under the square root in (5.11) is recognized as the squared norm of the vector $(\tilde{X}_\alpha, \nabla_\alpha \tilde{F})$ in the lagrangian submanifold $\mathbb{R} \times C$, invariant under the maximal compact subgroup of H. Expanding $S = S_1 - iS_2$ in powers of z, we may rewrite

$$S = \frac{\sqrt{|z|^6 + |z|^4 \sum_A x_A^2 + |z|^2 \sum_A [\partial_A I_3(x_A)]^2 + [I_3(x_A)]^2}}{|z|^2} - i\frac{x_0\, I_3(x_A)}{y|z|^2}\,. \tag{5.14}$$

In the limit $z \to 0$ with $I_3(x_A) > 0$, the spherical vector (5.10) therefore behaves as

$$\log f_K \sim \frac{I_3(x_A)}{yz} - \left(\nu + \frac{1}{2}\right) \log I_3(x_A) + \frac{1}{2} I_3(x_A) \sum_A [\partial_A I_3(x_A)]^2 + \mathcal{O}(|z|^2)\,. \tag{5.15}$$

Using the spherical vector f_K and a $G(\mathbb{Z})$-invariant distribution $\delta_{G(\mathbb{Z})}$ in $\mathcal{H}^*$, we may now construct an automorphic theta series as

$$\theta_G(g) = \langle \delta_{G(\mathbb{Z})}, \rho(g) \cdot f_K \rangle \tag{5.16}$$

where g takes value in $G(\mathbb{R})$ and $\rho(g)$ is the minimal unitary representation of $G(\mathbb{R})$ in $\mathcal{H}$ constructed above [39, 47]. Due to the invariance of f_K under the maximal compact subgroup K of G, the left-hand side is a well-defined function on $G(\mathbb{R})/K$, which is furthermore invariant under the arithmetic group $G(\mathbb{Z})$ — in other words, an automorphic form. Furthermore, the invariant distribution $\delta_{G(\mathbb{Z})}$ can be obtained by adelic methods, and is equal to the product over all primes p of the spherical vectors over the p-adic fields $\mathbb{Q}_p$ [47].

5.2 The minimal representation of E_8

Let us now spell out the above general construction for E_8 in more physical terms. E_8 is the U-duality group of type-II string theory compactified on T^7 (or M-theory compactified on T^8). Since black holes are static solutions in 4 dimensions, it is natural to consider black holes at finite temperature T, and think of the 4-th direction as a thermal circle of radius $R_0 = 1/T$. In the decompactification limit to 4 dimensions, E_8 decomposes into $E_7 \times \mathrm{Sl}(2)$, where the second factor is generated by $(E_{-\omega}, H_\omega, E_\omega)$. Accordingly, the moduli space in 3 dimensions factorizes into

$$\frac{E_8}{\mathrm{SO}(16)} = \frac{\mathrm{Sl}(2)}{\mathrm{U}(1)} \times \frac{E_7}{\mathrm{SU}(8)} \ltimes \mathbb{R}^{56} \tag{5.17}$$

where the last factor transforms as a 56 representation under E_7. Thus, there is a non-linear action of $E_8(\mathbb{R})$ on the 58-dimensional space $\mathrm{Sl}(2)/\,\mathrm{U}(1) \times \mathbb{R}^{56}$, by right multiplication on this decomposition (assuming that the fractions in (5.17) are left-cosets): this is the classical action of E_8 on the co-adjoint orbit of $E_{-\omega}$.[7] Using the general techniques[8] in [34], is is easy to understand the physical interpretation of these 58 variables: the first factor in (5.17) is described by

$$y + it = K_{0;01234567} + \frac{i R_0^2 V_{1234567}}{l_p^9} \tag{5.18}$$

while $\mathbb{R}^{56}$ is parameterized by two Sl(8) antisymmetric matrices[9] Q and P (equation (4.71) in [39], after flipping the last two rows and columns and relabelling R_8 into R_0)

$$Q = \begin{pmatrix} C_{0ij} & C_{0i7} & K^i \\ -C_{0i7} & 0 & K^7 \\ -K^i & -K^7 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} E_{0klmn7} & E_{jklmn7} & g_{0i} \\ -E_{jklmn7} & 0 & g_{07} \\ -g_{0i} & -g_{07} & 0 \end{pmatrix} \tag{5.19}$$

where i, j run from 1 to 6, and a dualization over $(j)klmn$ is understood. By decompactification of the thermal circle, the scalars g_{I0}, C_{IJ0}, E_{IJKLM0},$K_{I;IJKLMNP0}$ $(I, J, \dots = 1, \dots, 7)$

[7] By dropping the Cartan generator in $\mathrm{Sl}(2)/\,\mathrm{U}(1)$, one obtains the "quasi-conformal realization" of E_8 on 57 variables [42].

[8] In a nutshell [66]: represent the root lattice in a basis where the fundamental roots are $e_{i+1} - e_i$ $(i = 1, \dots 7)$ and $e_1 + e_2 + e_3 - e_0$ and associate to any vector $\alpha = \sum_{I=0}^{8} \alpha^I e_I$ the quantity $S = l_p^{3\alpha_0} \prod_{i=1}^{8} R_i^{\alpha_i}$, where l_p is the 11-dimensional Planck length; if α a positive root, S is the action of D-instanton conjugate to a Peccei-Quinn modulus in G/K.

[9] We use the same notation as in (2.2), but (2.2) and (5.19) are in fact conjugate to each other.

become gauge fields in 4 dimensions, which are precisely the 56 electric and magnetic gauge fields in (2.1). In fact, it is generally true that positive roots in the moduli space are conjugate to instantons, which become black holes in one dimension higher [34, 67]. This also allows to understand the meaning of y, t in (5.18): the imaginary part is the product of the inverse temperature square by the volume of the M-theory T^7 in Planck units. The real part is the scalar dual of the Kaluza-Klein gauge field $g_{0\mu}$ in 3 dimensions. Since the the off-diagonal $g_{0\mu}$ components are non-zero for spinning black holes, it is natural to think of y as a potential conjugate to the 4-dimensional angular momentum.

Now, in order to quantize this co-adjoint orbit, one should take a lagrangian subspace in $\mathbb{R}^{56}$. The standard polarization described in section 5.1 is obtained by Fourier transform over the last two columns (or rows) in Q, as well as x^0. Interpreting the direction 7 as the M-theory direction the "coordinates" in this polarization consist of the 1+27 potentials $x_0 = g_{07}, C_{ij0}, E_{ijklm0}, g_{0i}$ dual to the D0-brane, D2-brane, NS5-brane and Kaluza-Klein momentum on T^6. This is precisely the "large volume" polarization in (2.11). In this basis, the cubic invariant of E_6 entering the prepotential (5.2) is given by

$$I_3 = \mathrm{Pf}([D2]^{ij}) + \frac{1}{5!}\epsilon_{jklmnp}[kk]_i[D2]^{ij}[NS]^{klmnp} \tag{5.20}$$

where again we identify charges with their conjugate potentials. On the other hand, the Sl(8)-invariant polarization (5.19) can be reached by Fourier transform over the 13 variables $[kk]_i$ and $[NS]^{klmnp}$. The prepotential controlling the corresponding lagrangian submanifold is obtained by Legendre transform of (4.5) over the same variables, leading to

$$F_0^{\mathrm{Sl}(8)} = \sqrt{\mathrm{Pf}(Q)} \tag{5.21}$$

where Q is the antisymmetric 8×8 matrix in (5.19). This is a useful hint on the spherical vector f_{E_8} in the Sl(8) polarization, which is unknown until now [39].

5.3 Wigner function and spherical vector

In order to properly formulate our conjecture, let us return to (4.4): as noticed in [5], upon analytically continuing $\phi \to i\chi$, the left-hand side is interpreted as the Wigner function associated to the topological wave function $\Psi = e^F$:

$$\Omega(p^I, q_I) = \int d\chi\, \Psi^*(p^I - \chi^I)\Psi(p^I + \chi^I)\, e^{2\pi i \chi^I q_I}\,. \tag{5.22}$$

Now, let us postulate that the microscopic degeneracies $\Omega(p^I, q_I)$ are invariant under $G(\mathbb{Z})$ ($E_7(\mathbb{Z}) \subset G(\mathbb{Z})$ for M-theory on T^7), and investigate the consequences of this assumption for the wave function Ψ. For illustration purposes, we shall consider $G = \mathrm{Sl}(2,\mathbb{Z})$ acting on a single pair of conjugate charges (p, q) as a doublet. For the generator $q \to q + ap$, writing

$$\Omega(p, q + ap) = \int d\chi\, e^{-\frac{i\pi}{2}a(p-\chi)^2}\Psi^*(p-\chi)\, e^{\frac{i\pi}{2}a(p+\chi)^2}\Psi(p+\chi)\, e^{i\chi q} \tag{5.23}$$

the right-hand side is identified as the Wigner function of the transformed wave function

$$\tilde{\Psi}(p) = e^{\frac{i\pi a}{2}p^2}\Psi(p) = \Psi(p) + \frac{i\pi a}{2}p^2\Psi(p) + \mathcal{O}(a^2)\,. \tag{5.24}$$

Similarly, for an infinitesimal shift $q \to q + cp$, one may show by integration by parts that

$$\tilde{\Psi}(p) = \Psi(p) - \frac{ic}{8\pi}\,\partial_p^2\Psi(p) + \mathcal{O}(c^2) \tag{5.25}$$

Finally, under an exchange $(p,q) \to (-q,p)$, it is straightforward to check that $\Psi(p)$ is mapped to its Fourier transform. This means that, under a $\mathrm{Sl}(2,\mathbb{R})$ linear transformation of the phase space (p,q), the wave function $\Psi(p)$ transforms by a unitary representation of $\mathrm{Sl}(2,\mathbb{R})$ — to wit, the metaplectic representation. The $\mathrm{Sl}(2,\mathbb{Z})$ invariance of the microscopic degeneracies $\Omega(p,q)$ is thus equivalent to the invariance of Ψ under $\mathrm{Sl}(2,\mathbb{Z})$.

In this simple case, this problem has a well known solution, unique up to rescaling: $\Psi(p)$ is simply the "Dirac comb" distribution $\delta_{\mathbb{Z}}(p) = \sum_{m\in\mathbb{Z}}\delta(p-m)$. Indeed, since it is localized on the integers, it is invariant[10] under (5.24). It is also invariant under Fourier transform by the Poisson resummation formula. Recall furthermore that it can be obtained as a product over all primes p of the spherical vector of the metaplectic representation over $\mathbb{Q}_p$, which is the function equal to 1 for $x \in \mathbb{Z}_p$, 0 otherwise. Setting

$$\Psi(p) = \sum_{m\in\mathbb{Z}}\delta(p-m) \tag{5.26}$$

we find that the Wigner function is

$$\Omega(p,q) = \delta_{\mathbb{Z}}(2p)\delta_{\mathbb{Z}}(q) \tag{5.27}$$

which corresponds to a uniform distribution on the lattice of charges. Applying the prescription (5.16) in this case leads to the standard Jacobi theta series for $\mathrm{Sl}(2,\mathbb{Z})$ [47].

5.4 $\mathcal{N}=8$ black holes in 4D and the E_8 theta series

The lesson from the previous example is clear: assuming that the microscopic degeneracies $\Omega(p^I,q_I)$ in M-theory compactified on T^7 are indeed equal to the Wigner function of a wave function Ψ, the latter has to be invariant under a unitary representation of $E_7(\mathbb{Z})$ acting on the space of 28 variables p^I. Unfortunately, the minimal representation of E_7 has only functional dimension 17 (while the generic unitary representation of E_7, based on the coadjoint orbit of a generic diagonalizable element has functional dimension 61), and it does not appear likely that E_7 have a unirep of dimension 28 (although it does have a unirep of dimension 27 [42]). The minimal representation of E_8 however provides a natural unitary representation of E_7 on 28 variables, with an extra variable y, which is spectator under the action of E_7. Furthermore, the spherical vector for this representation over $\mathbb{Q}_p$ is known for all primes, providing a concrete $E_7(\mathbb{Z})$ (in fact $E_8(\mathbb{Z})$) invariant distribution $\delta_{E_8(\mathbb{Z})}$. We thus propose that the exact degeneracies (or rather, the helicity supertrace Ω_8) in M-theory compactified on T^7 are given by the Wigner transform of the distribution

[10] In fact, it is only invariant under (5.24) when $a \in 4\mathbb{Z}$; this is due to the fact that the metaplectic group is a 2-sheeted cover of $\mathrm{PSl}(2,\mathbb{Z})$. This subtlety does not occur when G is simply-laced.

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$\delta_{E_8(\mathbb{Z})}(y,p_A)$ in the (y,p_A,q^A) space. This proposal raises some interesting questions:

i) The computation in section 4.3 indicates that the classical limit of the Wigner function is effectively determined by the spherical vector f_K over $\mathbb{R}$ rather than $\mathbb{Q}_p$. It would be interesting to understand this in more detail.

ii) The spherical vector f_K has subleading corrections (5.15) to the prepotential (4.5) as $z \to 0$. Can one interpret them as higher-derivative corrections to the prepotential (4.5) ?

iii) The spherical vector is annihilated by the compact generators $E_\alpha \pm E_{-\alpha}$. Can we understand these partial differential equations, especially when α is the highest root, as a $\mathcal{N}=8$ version of the holomorphic anomaly equations ?

iv) One could in principle compute the Wigner function in the full phase space $(t,y,p^A,$ $q_A)$, which would be invariant under the full $E_8(\mathbb{Z})$ symmetry. Can this distribution be understood as a black hole partition function at finite temperature and angular potential?

We hope to return to these questions in a future publication.

5.5 $\mathcal{N}=4$ black holes in 4D and the D_{16} theta series

A similar reasoning can be applied in $\mathcal{N}=4$ models such as type-IIA string theory compactified on $K3\times T^2$, or its dual the heterotic string compactified on T^6. A counting function was proposed long ago in [31], based on an automorphic form of the modular group $\mathrm{Sp}(4,\mathbb{Z})$ of genus-2 Riemann surfaces,[11] and recently rederived using the 4D/5D connection in [32]. Compactifying down to 3D dimensions, the U-duality group $\mathrm{Sl}(2,\mathbb{Z})\times\mathrm{SO}(6,22)$ is enhanced to $\mathrm{SO}(8,24,\mathbb{Z})$, which decomposes as

$$\frac{\mathrm{SO}(8,24)}{\mathrm{SO}(8)\times\mathrm{SO}(24)} = \frac{\mathrm{Sl}(2)}{\mathrm{U}(1)} \times \left[\frac{\mathrm{Sl}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,22)}{\mathrm{SO}(6)\times\mathrm{SO}(22)}\right] \times \mathbb{R}^{56}\,. \tag{5.28}$$

Again, the last factor in (5.28) can be identified as the time component of the 28+28 electric and magnetic gauge fields in 4 dimensions, conjugate to the 28+28 electric and magnetic charges. It transforms linearly as a (2,28) representation of the 4-dimensional U-duality group. The first factor corresponds to the same field as in (5.18). By right multiplication, the 4-dimensional group acts symplectically on $\mathrm{Sl}(2)/\mathrm{U}(1)\times\mathbb{R}^{56}$, the coadjoint orbit of the lowest root of $\mathrm{SO}(8,24)$. The minimal representation of $\mathrm{SO}(8,24)$ is obtained by quantizing this orbit, and acts on functions of 29 variables: for the standard $\mathrm{SO}(6,22)$-invariant polarization, based on the prepotential

$$F_0 = \frac{X_1 X^a C_{ab} X^b}{X^0} \tag{5.29}$$

[11] In the proposal [31], the $\mathrm{SO}(6,22,\mathbb{Z})$ symmetry is realized trivially by including dependence on the square of the inner products of the charges $q_e^2, q_m^2, q_e\cdot q_m$ only. This may not be true when the charges have some common divisors.

where C_{ab} is a signature (5,21) quadratic form, these are the 28 electric charges in 4 dimensions, together with the variable y conjugate to the 4D angular momentum. It should be straightforward to adapt the $SO(16,16)$ spherical vector (5.13) to the $SO(8,24)$ real form.[12] The $SO(8,24,\mathbb{Z})$ invariant distribution $\delta_{D_{16}(\mathbb{Z})}$ may be computed as before by tensoring the spherical vectors over all p-adic fields computed in [68]. We thus propose that the micro-canonical degeneracies in the heterotic string compactified on T^6 are given by the Wigner function of the distribution $\delta_{D_{16}(\mathbb{Z})}$. It would be very interesting to understand the relation with the formula proposed in [31].

5.6 Conformal quantum mechanics

Finally, we would like to mention an interesting interpretation of the minimal representation, as the spectrum-generating symmetry of a a conformal quantum mechanical system [42, 48]. Consider the universal Sl(2) subgroup generated by $(E_\omega, H_\omega, E_{-\omega})$ in the standard polarization. Performing a canonical transformation [48]

$$y = \frac{1}{2}\rho^2\,, \qquad x_A = \frac{1}{2}\rho q_A \tag{5.30}$$

$$p = \frac{1}{\rho}\pi - \frac{1}{\rho^2} q_A \pi^A\,, \qquad p^A = 2\frac{\pi^A}{\rho} \tag{5.31}$$

the highest root generator E_ω becomes (up to computable ordering terms) the hamiltonian of a De Alvaro Fubini Furlan-type quantum mechanical system [69]:

$$E_{-\omega} = \frac{1}{2}\pi^2 + \frac{I_4(\pi^A, q_A)}{2\rho^2} \tag{5.32}$$

where $I_4(\pi^A, q_A)$ is given by the same expression (2.13) which related the black hole entropy to the cubic prepotential. The universal Sl(2) factor is interpreted as the conformal group in 0+1 dimensions, and is only part of the full E_8 spectrum generating symmetry. The spherical vector f_K may be viewed is the "most symmetric" state, which is as close to the ground state as one may hope to get for a hamiltonian whose spectrum is unbounded both from below and from above. It would be very interesting to understand the relation between this quantum mechanical system and the one controlling the cosmological / attractor flow of the moduli in the near-horizon geometry introduced in [14]. The conformal quantum mechanics (5.32) may also be related to the conformal models introduced in [70 – 72].

6. Conclusion

In this paper, we discussed the degeneracies of 4D and 5D BPS black holes in maximally supersymmetric compactifications of M-theory or type-II string theory, with U-duality as a powerful tool. Using the 4D/5D lift, we computed the exact degeneracies of 4D black holes with D0,D2 and unit D6 charge, and found agreement with the general expectation from U-duality at leading order. We also proposed a natural generalization of the 4D/5D lift

[12] The minimal representation of $SO(4,28)$ real form of D_{16} has been constructed recently in [64].

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to include all 56 charges of $\mathcal{N}=8$ supergravity in 4 dimensions. Utilizing the remarkable invariance of the prepotential under Legendre transform, we computed to leading order the "topological amplitude" which controls the $\mathcal{N}=8$ attractor formalism, and found an hint of a E_8 hidden symmetry in the black hole partition function. By analysing the physical interpretation of the minimal unipotent representation of E_8, we conjectured that exact BPS black hole degeneracies, with fixed angular momentum, should be given by the Wigner function of the unique $E_8(\mathbb{Z})$-invariant distribution in this representation. A similar conjecture relates the degeneracies of $\mathcal{N}=4$ black holes to the minimal representation of $\mathrm{SO}(8,24)$. The spherical vectors are known explicitely in both cases, and it would be very interesting to test these conjectures against other approaches such as [31, 32].

Another interesting question is the relation of the E_8 conformal quantum mechanics (5.32) which underlies the minimal representation with the radial/cosmological flow investigated in [14]: in particular, one would like to know if the E_8 conformal quantum mechanics (5.32) admits a supersymmetric extension, and if so, whether the truncation to BPS states is equivalent to the invariance under the maximal compact subgroup. If so, this would indicate that the "wave function of the Universe" in this mini-superspace formulation is indeed the spherical vector f_K, as suggested in [48].

Assuming that the admittedly speculative conjectures in this paper hold true, it is interesting to ask about the generalization to $\mathcal{N}=2$ supersymmetry. Several years ago, M. Kontsevitch made the "very wild guess" that the topological string amplitude should be an infinite dimensional solution to the "master equation" $\mathrm{Fourier}(e^F)=e^{\mathrm{Legendre}(F)}$ [73], of which the cubic prepotentials $F=I_3(X)/X^0$ which we encountered in this work are finite-dimensional solutions. Since the topological amplitude $\Psi=e^F$ can be thought of as a wave function in the topological B-model [74], it is indeed natural to expect that symplectic transformations on the Calabi-Yau periods will act by Fourier transform, and relate Gromov-Witten instanton series in different geometric phases. It is our hope that a careful study of the $\mathcal{N}=8$ case will help in making these ideas more precise.

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Note added: while this manuscript was being written up, a preprint appeared which independently derived the main result of section 3, while also providing an interesting microscopic rederivation [75]. I am grateful to A. Strominger for sending me a draft prior to publication. I also wish to thank M. Gunaydin for many helpful remarks on an earlier version of this manuscript, and for pointing out some important references.

A. E_7 minimal representation and black holes in 5 dimensions

By the same reasoning as above, one may also expect that the black hole partition function

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in 5 dimensions may be related to the minimal representation of E_7, since this is the U-duality group which appears under compactification on a thermal circle to 4 dimensions. The minimal representation of E_7 is based on the decomposition

$$\frac{E_7}{\mathrm{SU}(8)} = \frac{\mathrm{Sl}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,6)}{\mathrm{SO}(6)\times\mathrm{SO}(6)} \ltimes \mathbb{R}^{32}\,. \tag{A.1}$$

This is different from the decomposition $E_7 \to E_6 \times \mathbb{R}$ which controls the decompactification limit to 5 dimensions, and which is instead related to the "conformal" realization of E_7 on 27 variables [42]. Nevertheless, as we shall see, it may be sufficient to describe the Ramond-Ramond charges in 5 dimensions. Using the same techniques as before, we identify the last factor in (A.1) as the 16+16 Ramond-Ramond gauge fields and scalars in type-IIA on T^5 (where R_1 is the M-theory circle, $R_{2,3,4,5,6}$ are the radii of T^5 and R_7 is the radius of the 6th direction): in the SO(5,5) polarization (eq. (4.55) in [39]), the 5+10+1 "coordinates" correspond to the 5D scalars

$$Q = \{g_{1i}, C_{ijk}, C_{123456}\} \tag{A.2}$$

(where i,j,k run from 2 to 6) while the 1+10+5 "momenta" correspond to the reduction of the 5D RR vectors along the 6th direction,

$$P = \{g_{17}, C_{ij7}, C_{ijklm7}\}\,. \tag{A.3}$$

In addition, the Sl(2)/U(1) factor corresponds to $y + it = C_{234567} + iV_{234567}/l_p^6$. This is not the "standard" polarization of [39], which is invariant under Sl(6), the mapping class group of type IIB string theory compactified on the T-dual T^6. The latter can however be reached by a Fourier transform over the 5 variables $g_{17}, C_{237}, C_{247}, C_{257}, C_{267}$ [39].

While E_7 is expected to unify 5D black holes and 5D black strings [43], we find that the minimal representation of E_7 is unsuitable for this purpose, as it unifies 5D black holes and 5D instantons. Nevertheless, it may turn out to be relevant for 5D black degeneracies, in the following sense: E_7 admits a maximally commuting algebra of dimension 27, transforming as a 27 representation of E_6, which contains the SO(5,5) spinor Q in (A.2) as an isotropic vector (a "pure 27-sor" in the terminology of [47], i.e. a solution of the quadratic equations $27 \otimes 27|_{\overline{27}} = 0$). It is natural to conjecture that the Fourier coefficients of the E_7 theta series with respect to this commuting algebra may have a relation to the degeneracies of small black holes with zero tree-level entropy, i.e. solutions to the cubic equation $27^3|_1 = 0$.

B. E_6 minimal representation and black holes in 6 dimensions

Similarly, we expect that black hole degeneracies in 6 dimensions may have a hidden $E_6(\mathbb{Z})$ symmetry, larger than the naive U-duality group $\mathrm{SO}(5,5,\mathbb{Z})$.

The minimal representation of E_6 follows from the decomposition

$$\frac{E_6}{\mathrm{USp}(8)} = \frac{\mathrm{Sl}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{Sl}(6)}{\mathrm{SO}(6)} \ltimes \mathbb{R}^{20} \tag{B.1}$$

and acts on functions of 11 variables, which can be identified as

$$Q = \begin{pmatrix} 0 & C_{345} & C_{245} & C_{235} & C_{234} \\ & 0 & C_{145} & C_{135} & C_{134} \\ & & 0 & C_{125} & C_{124} \\ & a/s & & 0 & C_{123} \\ & & & & 0 \end{pmatrix}, \qquad y = E_{123456} \tag{B.2}$$

together with their conjugates

$$P = \begin{pmatrix} 0 & C_{126} & C_{136} & C_{146} & C_{156} \\ & 0 & C_{236} & C_{246} & C_{256} \\ & & 0 & C_{346} & C_{356} \\ & a/s & & 0 & C_{456} \\ & & & & 0 \end{pmatrix}, \qquad t = V_{123456}/l_p^6 \tag{B.3}$$

in the Sl(5)-invariant polarization ([39], eq. (4.45), after permuting a permutation (13)(45) on the rows and columns). This is related to the "standard" Sl(3) $\times$ Sl(3) invariant polarization, by prepotential $F_0 = \det(X)/X^0$, by Fourier transform over $C_{126}, C_{136}, C_{236}$. Choosing $R_6 = 1/T$ as the radius of the thermal circle, the variables P and y can be interpreted as the electric potentials dual to the $[M2]^{IJ}$ $(I, J = 2, \ldots 6)$ and $[M5]$ black hole charges, leaving no room for the $[KK]_I$ charges.

As in the E_7 case, the decomposition (B.1) does not preserve the U-duality symmetry SO(5, 5) in 6 dimensions.[13] Nevertheless, it can be checked that the 11 charges $[M2]^{IJ}$ and $[M5]$ transform as an isotropic vector of SO(5, 5) — in other words, a pure spinor of SO(5, 5), which satisfies $16 \otimes 16|_{\overline{16}} = 0$. It us thus tempting to conjecture that the Fourier coefficients of the E_6 theta series with respect to this dimension 16 abelian subalgebra are related to degeneracies of "small" black holes in 6 dimensions (indeed, all BPS black holes in 6 dimensions are "small", in that a smooth solution of the Einstein-Maxwell equations with the required charges does not exist [76, 77]). It would be very interesting to understand the relation with the approach in [49, 50].

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